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VERTEX LABELING OF A HALF-CUBE TO INDUCE BINARY FACE LABELING

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ABSTRACT: A plane, passing through the center and orthogonal to a diagonal, slices a cube into two identical halves each having three triangles, three pentagons and a hexagon. If you label the ten vertices of a half-cube with numbers, then each face is said to receive an induced face label given by the sum of all vertices around it. Label the ten vertices of a half-cube with digits 0 through 9 so that the induced labels of the three triangles and the three pentagons constitute two distinct values.

Keywords: Puzzle, solution, rotation, reflection, permutation.

AMS Subject Classification: 05C78

1. Introduction

Having studied a symmetric random walk on the vertices of a cube (see [1]), we were looking around for other simple objects derived from a cube. A pleasant surprise awaited us when we attended a neighborhood pitch-in tea party: We noticed that the hostess had served cheese cubes cut in halves. She explained that two guests had volunteered to bring cheese cubes; but earlier in the day one of them called in sick; so, she cut the cubes brought in by the other guest. As she sliced the cubes in half, with plane cuts passing through the center, she playfully tilted the cuts at various angles. The resulting half-cubes, with their curious shapes, served not only to satisfy the guests' appetite, but also as icebreaker to start conversation.

Later we asked: "How many topologically distinct half-cubes are there?" The answer is four—with 6, 7, 8 and 10 vertices, respectively. We leave the proof to the astute reader. The half-cube with 8 vertices is isomorphic to the cube; the other three are called rectangular-, rhombic- and hexagonal half-cubes according as the shape of the new face generated by the plane cut. We will study random walks on these half-cubes in due time. In this paper, we will focus on labeling the vertices of a hexagonal half-cube. Motivated readers may wish to study the other half-cubes.

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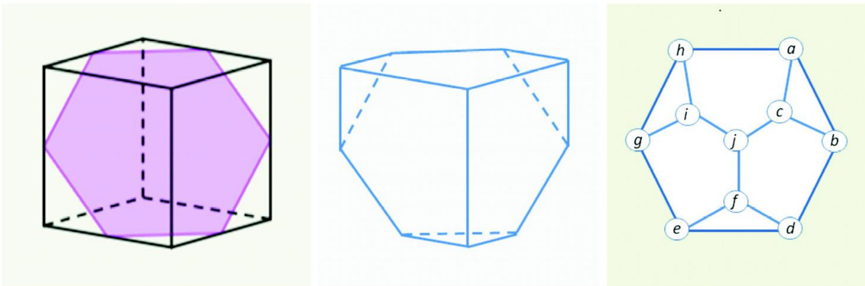

Figure 1. A cube, a half-cube, and the planar graph of a half-cube labeled a through j

When a cube is cut by a plane passing through the center and orthogonal to a diagonal, we obtain two identical hexagonal half-cubes. Each half-cube has 10 vertices, 15 edges and 7 faces—three (isosceles, right) triangles, three pentagons and one (regular) hexagon. Figure 1 shows a cube cut in half (the diagram is taken from [2]), a half-cube, and its planar graph with vertices labeled with letters a through j. (A lowercase letter represents a number; the corresponding uppercase letter names the vertex.) Throughout the paper, we abide by this ordering convention of the ten vertices.

Motivated by the geometric symmetry in Figure 1, in [3] we posed and solved a curious puzzle, which goes as follows:

Puzzle 1:

Label the ten vertices of the half-cube with digits 0, 1, ..., 9 (using each digit only once) in such a way that the induced labeling of the three triangles given by the sum of their three vertices is a constant (the T-constant property), or the induced labeling of the three pentagons given by the sum of their five vertices is a constant (the P-constant property), or both (the TP-constant property).

In [3], we proved that if we do not distinguish labeling that are equivalent after rotation, reflection, or complementation (that is, subtraction from 9), then there are 864 T-constant labeling, 544 P-constant labeling, and a *unique* TP-constant labeling, each of which gives rise to 12 distinct labeling when we do allow rotation, reflection, and complementation. In fact, for a TP-constant labeling, either the triangles add up to T = 13 and the pentagons to P=25, or the triangles add up to T=14 and the pentagons to P=20. None of these 12 solutions exhibits equal sums for all six faces, proving that it is impossible for all six face sums (other than the hexagon) to be equal.

In this paper, we focus on those labeling which cause the six face labels take on two distinct values repeated as often as necessary. Specifically, we ask the following questions:

Q1. Can five faces receive a constant sum, leaving the sixth face label to be different?

Q2. Can four faces receive a constant sum, and the other two faces another constant sum?

Q3. Can two triangles and a pentagon achieve the same sum, while the third triangle and the remaining two pentagons achieve the other sum?

Solutions to Questions Q1–Q3 (and to Puzzle 1) are called induced binary labeling of the six faces since their sums take on two distinct values. In this paper, we answer these questions by listing all possible labeling up to rotation and reflection symmetries. (Complementation symmetry applies to Puzzle 1, but not to Questions Q1–Q3). We encourage readers to solve them on their own or in collaboration with others before reading this paper further. As an aid, we give them a template in Figure 2. We also invite astute readers to pose and solve other interesting labeling puzzles.

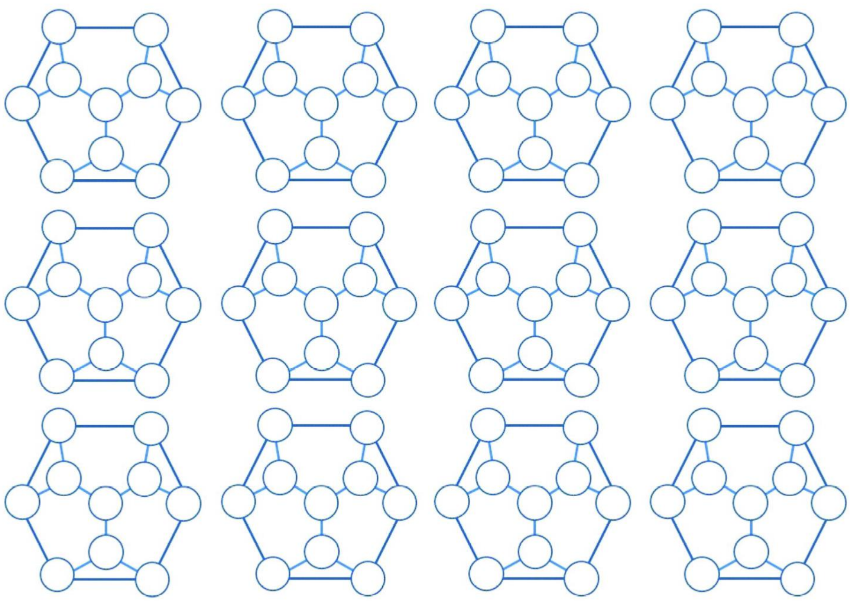

Figure 2. A template to label the vertices of a hexagonal half-cube

2. Five Face Sums Equal, One Face Sum Different

If five face sums are equal, the sixth face, with a different sum, must be either (1) a pentagon, or (2) a triangle. Thus, we have two sub-problems to solve.

Answering Question Q1 in Case 1

Let the five face sums be K and the sixth face (a pentagon) sum be $L \neq K$. Without loss of generality, let the sixth face be pentagon $ACJIH$. The three triangle sums add to $3K$ implying that $j = 45 - 3K$, and as such 3 divides j. Hence, the pair (j, K) is one of the following four pairs (0, 15), (3, 14), (6, 13), (9, 12). For these four pairs, by subtracting the sums of triangle GHI and pentagon $BCJFD$ from the sum 45 of all digits 0 through 9, and by subtracting the sums of triangle ABC and pentagon $EFJIG$ from 45, we note that $a + e = d + h = 45 - 2K$ take on values 15, 17, 19, 21, respectively. The last two cases are impossible since no two digits add up to 19 or 21. The second case is also impossible because there is only one way for two digits to add to 17, namely 8 and 9. Therefore, both $a + e$ and $d + h$ cannot be 17.

Therefore, if a solution exists in this case of Question 1, then we must have $j = 0$, $K = 15$. Comparing triangle ABC and pentagon $BCJFD$, we have $a = j + f + d = f + d$, and comparing triangle DEF and pentagon $EFJIG$, we have $d = j + i + g = i + g$. Hence, we have

$$15 = b + c + a = b + c + f + i + g$$

Since $j = 0$ is already used up, if five digits between 1 and 9 add up to 15, we know what those digits are (though we may not know the order); that is, we know $\{b, c, f, i, g\} = \{1, 2, 3, 4, 5\}$. Next, by comparing adjacent triangle and pentagon, we express the remaining four digits a, d, e, h in terms of the first five digits as follow:

$$d = g + i,\ a = f + d,\ e = b + c,\ h = f + e$$

Of course, these four distinct numbers d, a, e, h must constitute $\{6, 7, 8, 9\}$, implying that either ($f = 1$, $e = b + c = 6$, $d = g + i = 8$) or ($f = 2$, $e = b + c = 6$, $d = g + i = 7$). In the former case, $f = 1$, $\{b, c\} = \{2, 4\}$, $\{g, i\} = \{3, 5\}$; in the latter, $f = 2$, $\{b, c\} = \{1, 5\}$, $\{g, i\} = \{3, 4\}$. For each of $f = 1$ and $f = 2$, by assigning the pairs $\{b, c\}$ and $\{g, i\}$ in all four possible ways, we get exactly 8 labeling that make five face sums 15 and the sixth face sum larger (taking values 21, 22, 23, 25, 26 with respective frequencies 2, 1, 2, 2, 1). These 8 solutions are shown in Figure 3. Note that each solution exhibits $L > K$. However, it should come as no surprise since $5K + L = 90 + c + f + i + j$ and $K = 15$ imply that $L = 15 + c + f + i + j \geq 21$.

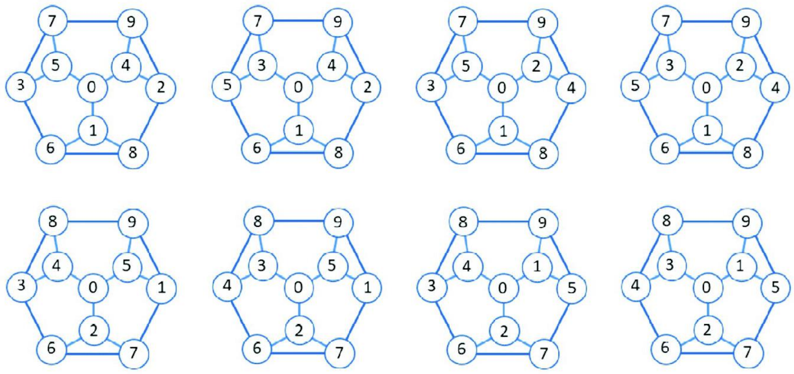

Figure 3. The 8 distinct labeling (not counting rotation and reflection symmetries) with five face sums 15 and the sixth (pentagonal) face sum larger (21, 22, 23, 25, or 26)

Answering Question Q1 in Case 2

Let the five face sums be K and the sixth face (a triangle) sum be $L \neq K$. Without loss of generality, let the sixth face be triangle ABC. The pentagon sums add up to $3K$, from which if we subtract the sum of all digits 0 through 9, we get $c + f + i + 2j = 3K - 45$. Also, adding up all six face sums, we have $5K + L = 90+c + f + i + j$, whence $6 \leq c + f + i + j \leq 30$ implies that $96 \leq 5K + L \leq 120$, or $96 - 5K \leq L \leq 120 - 5K$. Next, $6 \leq b+g+a+e \leq 30$ implies that $b+g=45-2K = a+e$ belongs to [3, 15], whence $15 \leq K \leq 21$. Hence, $3K \geq 45$ and $L = 45 - 2K - j \leq 15 - j \leq K$. Moreover, $L \neq K$ implies that $L < K$. For each K in [15, 21], keeping in mind the above-mentioned inequalities, let us search for viable values of j, $L = a + b + c = 45 - 2K - j$ and $c + f + i = 3K - 45 - 2j$ in the table below. Only three cases marked in the last column as X, Y, and Z satisfy all inequalities, deserving further study. Note that in all three cases, $j = 0$.

In Case Z, the two pentagons adjacent to the triangle with induced label 9 cannot both have sum 18 since the solution to $1 + 3 + 0 + 4 + z = K = 18$, namely $z = 10$, is not a permissible digit. In each of cases X and Y, by permuting the pair of peripheral vertices in the triangle ABC with sum 11, we find exactly one solution. These two solutions are shown in Figure 4.

3. Four Face Sums Equal, Other Two Face Sums Also Equal

Suppose that four face sums are equal, and the remaining two face sums are also equal but different from the first four. First, we shall show that

(1) if the two faces with equal labels consist of pentagons, then they have a *larger* label than the first four faces;

Table 1: Search for viable values of j, $a + b + c$, $c + f + i$

		$3 \le L < K$	$L + j =$	$L =$	j	$3K - 45 - 2j =$			
K	$[96 - 5K, 120 - 5K]$	$L \in$	$45 - 2K$	$a + b + c$	j	$c + f + i$	$\{a + b\} + c$	$c + \{f + i\}$	Casesto study
15	[21, 45]	none							
16	[16. 40]	none							
17	[11, 35]	[11, 16]	11	11	0	6	$\{4 + 6\} + 1$	$1 + \{2 + 3\}$	X
							$\{4 + 5\} + 2$	$2 + \{1 + 3\}$	Y
18	[6, 30]	[6, 17]	9	9	0	9	$\{1 + 5\} + 3$	$3 + \{2 + 4\}$	Z
				8	1	7	$0 + 3 + 5$	none	
				7	2	5	$0 + 3 + 4$	none	
				6	3	3	none		
19	[3, 25]	[3, 18]	7	7	0	12	$1 + 2 + 4$	none	
				6	1	10	$0 + 2 + 4$	none	
				5	2	8	$0 + 1 + 4$	none	
				4	3	6	none		
				3	4	4	$0 + 1 + 2$	none	
20	[3, 20]	[3, 19]	none						
21	[3, 15]	[3, 20]	none						

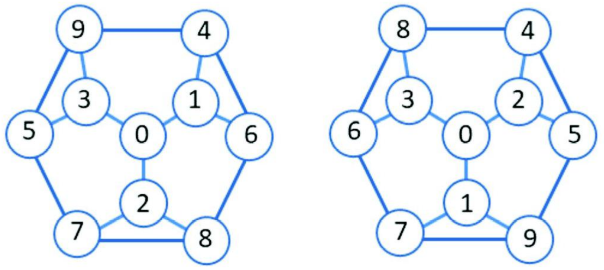

Figure 4. Two labeling with five face sums equal (17) and the sixth (triangular) face sum smaller (11)

(2) if the two faces with equal labels consist of triangles, then they have a *smaller* label than the first four faces; and

(3) it is *impossible* for the set of two faces with equal labels to consist of a triangle and a pentagon (whether the two faces are opposite or adjacent to each other).

Second, we shall find the solutions to the above cases (1) and (2).

Proofs of the above Claims for Cases (1)-(3):

(1) Suppose that two pentagons are labeled L and the other four faces are labeled K. In this case, all three triangles are labeled K, implying that and $3K + j = 45$ and $K \leq 15$. However, adding up all six face labels, we have $4K + 2L = 90 + c + f + i + j$. If $L \leq K$, then $90 + c + f + i + j \leq 6K$, implying that $K \geq 15 + (c + f + i + j)/6 \geq 16$. Thus, we reach a contradiction. Hence, $L > K$.

(2) Suppose that two triangles are labeled K and the other four faces are labeled L. In this case, all three pentagons are labeled L, implying that $3L = 45 + c + f + i + 2j \geq 51$ or $L \geq 17$. Here we have used $c + f + i + j \geq 6$. However, adding up all six face labels, we have $2K + 4L = 90 + c + f + i + j$. If $L \leq K$, then $90 + c + f + i + j \geq 6L \geq 102$, implying that $c + f + i + j \geq 12$. Next, by repeating the above inequalities, we get $3L = 45 + c + f + i + 2j \geq 57$ or $L \geq 19$; and $90 + c + f + i + j \geq 6L \geq 114$, implying that $c + f + i + j \geq 24$. Finally, repeating the inequalities once more, we get $3L = 45 + c + f + i + 2j \geq 69$ or $L \geq 23$; and $90 + c + f + i + j \geq 6L \geq 138$, implying that $c + f + i + j \geq 48$. However, this last inequality is impossible since $c + f + i + j \leq 9 + 8 + 7 + 6 = 30$. Thus, we reach a contradiction. Hence, $L > K$.

(3) Suppose that triangle DEF and its opposite pentagon $ACJIH$ are labeled L and the remaining four faces are labeled K. Then comparing triangle ABC and pentagon $BCJFD$, we have $a = d + f + j$; likewise, comparing

triangle HGI and pentagon $GIJFE$, we have $h = e + f + j$. Adding these two equalities, we see that

$$a + h = d + e + 2f + 2j > d + e + f = L = a + c + j + i + h > a + h$$

which is a contradiction. Next, suppose that triangle DEF and an adjacent pentagon $BCJFD$ are labeled L and the remaining four faces are labeled K. Then comparing triangle DEF and pentagon $BCJFD$, we have $e = b + c + j$; comparing triangle HGI and pentagon $GIJEF$, we have $h = e + f + j$; and comparing triangle ABC and pentagon $ACJIH$, we have $b = h + i + j$. Combining these three equalities, we see that

$$b = h + i + j = e + f + i + 2j = b + c + f + i + 3j \geq 10,$$

which is a contradiction since we can label the vertices with digits 0 through 9 only.

Hence, it is **impossible** for a triangle and a pentagon to have the same label, while the remaining four faces have another constant label.

This completes the proofs of the claims (1)–(3).

Next, we shall find the solutions to Question Q2 in Cases (1) and (2).

Answering Question Q2 in Case (1)

Suppose that three triangles and one pentagon have sum K each and the other two pentagons have sum $L > K$ each. Without loss of generality, let these two pentagons be $BCJFD$ and $EFJIG$. Then $j = 45 - 3K$ in $[0, 9]$ implies that j is a multiple of 3 and $12 \leq K \leq 15$. For these values of K, the central vertex takes values $j = 45 - 3K = 9, 6, 3, 0$, respectively; and the sum of vertices B and G takes values $b + g = 45 - 2K = 21, 19, 17, 15$, respectively. However, $1 \leq b + g \leq 17$, and without loss of generality, we can assume $b < g$. Therefore, either

(i) $K = 14, j = 3, b = 8, g = 9$, or

(ii) $K = 15, j = 0, b = 7, g = 8$, or

(iii) $K = 15, j = 0, b = 6, g = 9$.

For each of these three viable choices of (K, j, b, g), there are four ways to assign a, c, h, i, each leading to a unique assignment of d, e, f to ensure the remaining pentagon sums are equal; that is, $b + c + j + f + d = g + i + j + f + e$. Of the 12 assignments, 6 are solutions to Question Q2 and are shown in Figure 5.

Answering Question Q2 in Case (2)

Suppose that three pentagons and one triangle have sum K each and the other two triangles have sum $L < K$ each. Without loss of generality, let these two triangles be ABC and GHI. Then adding up the pentagon sums,

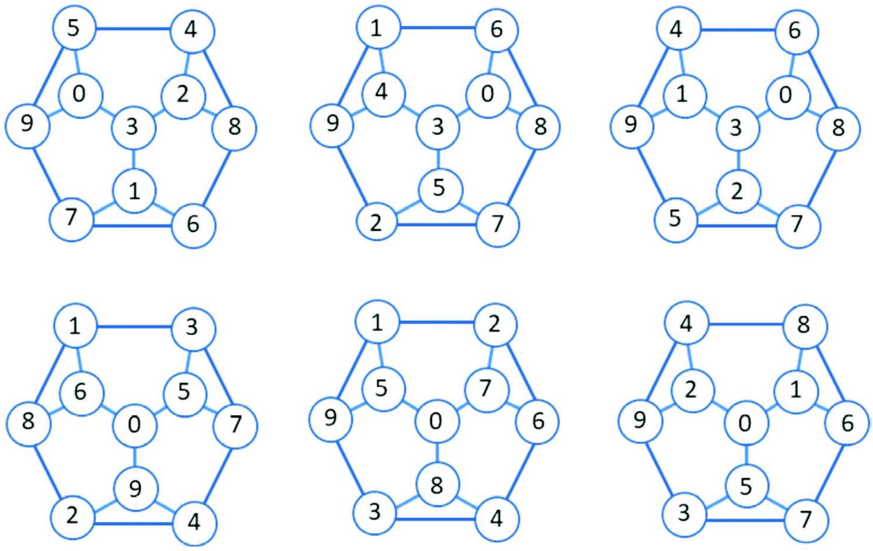

Figure 5. Six labeling with three triangle sums and one pentagon sum equal and the remaining two pentagon sums equal and larger

we have $3K = 45 + c + f + i + 2j$, whence $c + f + i + 2j = 3K - 45$ which is in [6, 39] implying that $17 \leq K \leq 28$. Adding up the triangle DEF and pentagon $ACJIH$ we have $b + g = 45 - 2K$ which is in [1, 17], implying that $28 \leq 2K \leq 44$ or $14 \leq K \leq 22$. Combining the two ranges for K we have $17 \leq K \leq 22$.

Adding up the triangle sums, we have $K + 2L + j = 45$; and adding up all six face sums, we have $4K + 2L = 90 + c + f + i + j$. We shall consider all combinations (K, j) for $17 \leq K \leq 22$ and $0 \leq j \leq 9$; then we shall compute $b + g = 45 - 2K$, $L = (45 - K - j)$ and $c + f + i = 4K + 2L - 90 - j$. Moreover, we shall assume, without loss of generality, that $b < g$.

In Table 2, we consider these combinations; check for the viability condition for (K, j). Note that if (K, j_0) is not viable (marked N), then so are (K, j) for all $j \geq j_0$. For the viable combinations, we proceed with labeling until we reach a contradiction (marked X), or we find one solution (marked Y) or two solutions (marked $Y2$). The seven solutions are shown in Figure 6.

Three Mixed-type Face Sums Equal, the Other Three Face Sums Equal

In [3], we proved that three triangle sums are equal, and three pentagon sums are equal for a *unique* labeling up to rotation, reflection, and complementation symmetry. Here we ask, can the two distinct face sums be

Table 2. For all (K, j) combinations, we compute $b + g = 45 - 2K$, then report $(L, c + f + i)$ and declare whether the combination is viable and if so, with how many solutions (N = not viable, X = viable but with no solution, Y = viable with one solution, $Y2$ = viable with two solutions)

$(L, c+f+i)$, viable?		j								
K	$b+g$	0	1	2	3	4	5	6	7	8
17	11	(14,6), X		(13,2), N						
18	9		(13,7), X		(12,3), $Y2$		(11,–1), N			
19	7	(13,12), $Y2$		(12,8), X		(11,4), Y		(10,0), N		
20	5		(12,13), X		(11,9), X		(10,5), $Y2$		(9,1), N	
21	3	(12,18), X		(11,14), X		(10,10), X		(9,6), X		(8,2), N
22	1		(11,19), X		(10,15), X		(9,11), X		(8,7), N	

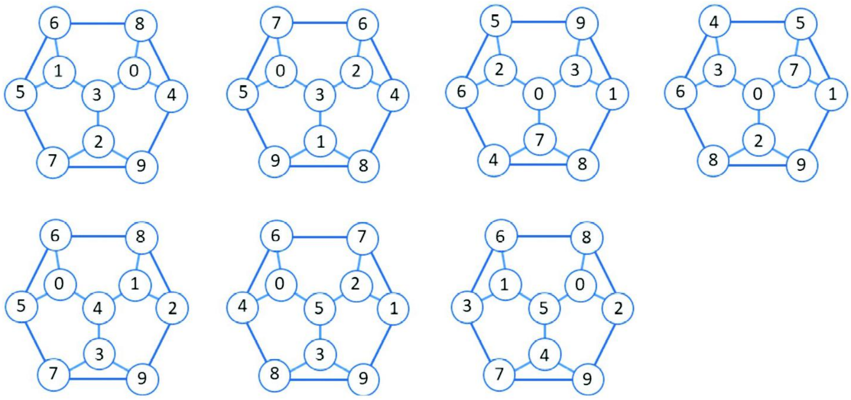

Figure 6. Seven labeling with three pentagon sums and one triangle sum equal and the remaining two triangle sums equal and smaller

achieved by mixed-type faces? That is, can two triangles and a pentagon attain a common sum while the remaining three faces attain a different common sum?

Surprisingly, the answer is affirmative when both sets of three faces are contiguous (two faces are contiguous if they share a common edge), but negative otherwise. That is, starting from a triangle and going clockwise, it is possible to achieve face sums K, K, K, L, L, L with $K < L$; that is, the face sum is smaller for the set consisting of two triangles and one pentagon and larger for the set consisting of two pentagons and one triangle. We simply report the complete list of nine solutions (up to rotation and reflection symmetries) in Figure 7. However, it is *impossible* to achieve face sums K,

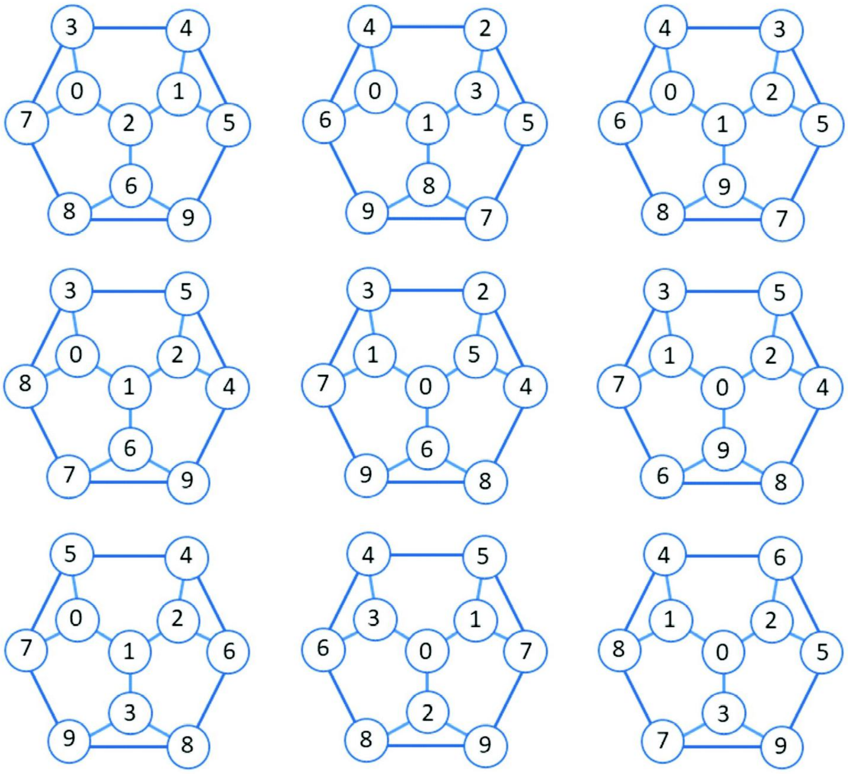

Figure 7. Nine distinct labeling with two sets of three contiguous faces adding up to a constant

L, K, K, L, L with $K \neq L$. We leave to the reader to prove these claims along the lines shown in the previous sections; or they can see [4] for details.

4. Summary and Conclusion

In this paper we have listed all solutions to labeling the ten vertices of a half-cube with digits 0 through 9 that induce binary labeling to all six faces (other than the hexagon). Up to rotation and reflection symmetries, the 34 solutions are summarized in Table 3, after rotating and reflecting to ensure that $c < f < i$. When we permit rotation and reflection, each solution leads to 6 permutations that are also solutions to Questions Q1–Q3.

We fondly hope readers will pose and solve other graph labeling problems using the cube, any one of the three half-cubes, or any other nice graph. See, for example, [5]–[7]. We highly recommend consulting [8] to check if a proposed labeling has been documented already. We conclude with a gentle

Table 3. A complete list of all solutions to Questions Q1–Q3 up to rotation and reflection symmetries. Without loss of generality, we assume that $c < f < i$. Each listed solution leads to 6 solutions when rotation and reflection are allowed (so that (c, f, i) are permuted)

Description	*K*	*L*	*Solutions a through j*	*Description*	*K*	*L*	*Solutions a through j*
Three triangles labeled the same, three pentagons labeled the same (TP-constant)	13	25	571-328-049-6	Three contiguous faces labeled *KKK*, other three contiguous faces labeled *LLL*	10	24	640-253-798-1
	14	20	590-761-248-3		10	24	640-352-789-1
					10	23	730-451-986-2
					11	23	731-245-896-0
					11	23	731-542-869-0
					11	22	830-542-976-1
					12	20	750-462-893-1
					13	19	571-982-643-0
					13	19	841-652-973-0
Four larger faces, two smaller faces (triangles)	10	20	460-712-983-5	Four smaller faces, two larger faces (pentagons)	14	20	590-761-842-3
	10	20	280-631-794-5		14	20	860-491-572-3
	11	19	560-821-973-4		14	23	860-194-275-3
	12	18	480-651-792-3		15	19	681-492-375-0
	12	18	750-981-462-3		12	25	735-186-249-0
	13	19	982-643-517-0		15	25	915-267-438-0
	13	19	652-913-847-0				
Five larger faces, one smaller face (triangle)	11	17	791-542-863-0	Five smaller faces, one larger face (pentagon)	15	21	681-492-753-0
	11	17	461-872-593-0		15	21	951-762-483-0
					15	21	951-762-384-0
					15	23	681-492-735-0
					15	23	861-573-924-0
					15	25	681-294-735-0
					15	25	762-483-915-0
					15	26	762-384-915-0

reminder that while high school students and even some middle school students may be given such problems as mini puzzles to solve for fun, compiling a complete list of all solutions or proving that no solution exists remains a challenge worthy of attention by reputed mathematicians.

Acknowledgment

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